JOURNAL OF COMPUTATIONAL PHYSICS 65, 120–137 (1986)

The Fourier Pseudospectral Method with a Restrain Operator for the Korteweg–de Vries Equation

MA HE-PING AND GUO BEN-YU

Department of Mathematics, Shanghai University of Science and Technology, Shanghai, China

Received December 28, 1984; revised October 16, 1985

In this paper we develop a Fourier pseudospectral method with a restrain operator for the Korteweg–de Vries equation. We prove the generalized stability of the schemes and give convergence estimations depending on the smoothness of the solution of the P.D.E. © 1986 Academic Press, Inc.

Recent publications in spectral methods for nonlinear partial differential equations provide a new potent solution technique (see [1–9]). In many of the relevant papers, pseudospectral methods are used, because they are more efficient than spectral methods (see [10–15]). But sometimes pseudospectral methods have a nonlinear instability which causes an anomalous increase of energy, or weakens the nonlinearity of the solution. In order to eliminate these phenomena, filtering or smoothing techniques are used (see [16–18]).

In this paper a restrain operator R is used to develop a semi-discrete or fully discrete Fourier pseudospectral method for the Korteweg–de Vries (K.d.V.) equation. Generalized stability and convergence estimates depending on the smoothness of the solution of the P.D.E. are proved.

I. THE SCHEMES

Consider the K.d.V. equation with periodic boundary condition:

$$\begin{aligned} \partial_t u + u u_x + u_{xxx} &= 0, & -\infty < x < \infty,\ 0 \leqslant t \leqslant T, \\ u(x+1, t) &= u(x, t), & -\infty < x < \infty,\ 0 \leqslant t \leqslant T, \\ u(x, 0) &= u_0(x), & -\infty < x < \infty. \end{aligned} \tag{1.1}$$

Let $x \in I = (0, 1)$, $L^2(I)$ with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. For any positive integer n, the semi-norm and the norm of $H^n(I)$ are denoted by $|\cdot|_n$ and $\|\cdot\|_n$, respectively. Let $C^{\infty}_{(p)}(I)$ be the set of infinitely differentiable functions with

0021-9991/86 $3.00

period 1, defined on R. $H^n_{(p)}(I)$ is the closure of $C^\infty_{(p)}(I)$ in $H^n(I)$. For any real $\sigma > 0$, define $H^\sigma_{(p)}(I)$ by complex interpolation between $H^{[\sigma]}_{(p)}(I)$ and $H^{[\sigma]+1}_{(p)}(I)$ where $[\sigma]$ denotes the largest integer which is smaller than σ. We have

$$H^\sigma_{(p)}(I) = \left\{ U \in L^2(I) \,\middle|\, \sum_{k=-\infty}^{\infty} (1+|k|)^{2\sigma} \, |\hat{U}_k|^2 < \infty,\ U(x) = U(x+1) \right\}$$

where $\hat{U}_k = (U, e^{2\pi i k x})$.

Let A be a Banach space. $C(0, T; A)$ is a set of strongly continuous functions from $[0, T]$ to A and $L^2(0, T; A)$ is a set of strongly measurable functions $u(t)$ from $(0, T)$ to A satisfying

$$\|u\|_{L^2(0,T;A)} = \left(\int_0^T \|u(t)\|_A^2 \, dt \right)^{1/2} < \infty.$$

Other similar notations have the usual meanings.

For any positive integer N, set

$$v_N = \operatorname{span}\{ e^{2\pi i k x} \mid |k| \leqslant N \}$$

and let $\dot{v}_N$ be a subspace of v_N of real-valued functions.

Let $h = 1/(2N+1)$ be the mesh size in variable x and $x_j = jh$ $(j = 0, 1, ..., 2N)$. The discrete inner product and norm are defined by

$$(u, v)_N = h \sum_{j=0}^{2N} u(x_j) \overline{v(x_j)}, \qquad \|u\|_N = (u, u)_N^{1/2}.$$

Let $P_N \colon L^2(I) \to v_N$ be the orthogonal projection operator, i.e.,

$$(p_N u, \varphi) = (u, \varphi), \qquad \forall \varphi \in v_N, \tag{1.2}$$

and $p_C \colon C(\bar{I}) \to v_N$ be the interpolation operator such that

$$p_C u(x_j) = u(x_j), \qquad 0 \leqslant j \leqslant 2N. \tag{1.3}$$

For any $u, v \in C(\bar{I})$, we can prove [16]

$$(u, v)_N = (p_C u, p_C v)_N = (p_C u, p_C v). \tag{1.4}$$

Approximation to (1.1) by Fourier pseudospectral methods directly needs the estimation

$$|(u u_x, w)_N| = |(p_C(u u_x), p_C w)| \leqslant C \|u\|_N^2 = C \|u\|^2, \qquad \forall u \in v_N, \tag{1.5}$$

where C is a constant depending only on w in v_N, in order to get the stability and convergence in L^2-norm. But (1.5) may not be true. As an example, we consider the functions

$$w = 1 - 2\sin 2\pi x = 1 + i(e^{2\pi i x} - e^{-2\pi i x}),$$

$$u = \sum_{|k| \leqslant N} a_k e^{2\pi i k x},$$

and obtain from (1.4)

$$(uu_x, w)_N = (p_N(uu_x), w) + ((p_C - p_N)(uu_x), w).$$

Since

$$|(p_N(uu_x), w)| = |(uu_x, w)| = \tfrac{1}{2}|(u^2, w_x)| \leqslant C\|w_x\|_{L^\infty} \|u\|^2$$

and

$$\begin{aligned} |((p_C - p_N)(uu_x), w)| &= \left|\left(\sum_{|k| \leqslant N} \sum_{\substack{|l| \leqslant N \\ |k-l| > N}} 2\pi i l a_l a_{k-l} e^{2\pi i k x}, w\right)\right| \\ &= |2\pi i\{N a_N a_{-1-N}\overline{(-i)} + (-N)\, a_{-N} a_{1+N} \bar{i}\}| \\ &= 2\pi N |a_N^2 + a_{-N}^2|, \end{aligned}$$

(1.5) can not be true. Obviously the trouble is only due to the higher frequencies. So we use the operator $R = R(\gamma)$ defined below to improve the scheme.

Guo Ben-yu [7, 8] pointed out that a better result can be obtained in solving numerically P.D.E. by using the generalized Fourier method (see [19]). Let $\gamma \geqslant 1$ and

$$u = \sum_{|k| \leqslant N} a_k e^{2\pi i k x},$$

we define $R = R(\gamma)$ by

$$Ru = \sum_{|k| \leqslant N} \left(1 - \left(\frac{|k|}{N}\right)^\gamma\right) a_k e^{2\pi i k x}. \tag{1.6}$$

In order to approximate the nonlinear term uu_x reasonably we define the operator $J_C: v_N \times v_N \to v_N$ as

$$J_C(u, v) = \tfrac{1}{3} p_C(u_x Rv) + \tfrac{1}{3}(p_C(uRv))_x.$$

If u, v, and $w \in \dot{v}_N$, then we have from (1.4)

$$(J_C(u, v), w) + (J_C(w, v), u) = 0. \tag{1.7}$$

The semi-discrete pseudospectral method for the problem (1.1) is to find u_C in $\dot{v}_N$ such that

$$\begin{aligned} \partial_t u_C + J_C(u_C, u_C) + u_{Cxxx} &= 0, \qquad -\infty < x < \infty,\ t > 0, \\ u_C(x, 0) &= p_C u_0(x), \qquad -\infty < x < \infty. \end{aligned} \tag{1.8}$$

By (1.7), the solution of (1.8) for all $t \geqslant 0$ satisfies

$$\|u_C(t)\| = \|u_C(0)\|.$$

Let τ be the mesh size in variable t. Denote $u^k(x) = u(x, k\tau)$ by u^k. Define

$$u_t^k = \frac{1}{\tau}(u^{k+1} - u^k)$$

and

$$|||u|||_\sigma = \max_k \|u^k\|_\sigma.$$

The fully discrete pseudospectral method for problem (1.1) is to find u_c^k in $\dot{v}_N$ such that

$$\begin{aligned} u_{ct}^k + J_C(u_c^k + \delta_1 \tau u_{ct}^k, u_c^k) + u_{cxxx}^k + \delta_2 \tau u_{cxxxt}^k &= 0, \qquad -\infty < x < \infty,\ k \geqslant 0, \\ u_c^0 &= p_C u_0, \qquad -\infty < x < \infty. \end{aligned} \tag{1.9}$$

If $\delta_1 = \delta_2 = \frac{1}{2}$, then

$$\|u_c^k\| = \|u_c^0\|, \qquad \forall k \geqslant 0.$$

II. The Main Theoretical Results

First consider the generalized stability of scheme (1.8) (the definition can be found in [20]). If u_C and the right term in (1.8) have errors $\tilde{u}$ and $\tilde{f} \in \dot{v}_N$, respectively, then

$$\partial_t \tilde{u} + J_C(\tilde{u}, u_C + \tilde{u}) + J_C(u_C, \tilde{u}) + \tilde{u}_{xxx} = \tilde{f}. \tag{2.1}$$

THEOREM 1. *If $\varepsilon > 0$, then there exists a positive constant C depending on $\|u_C\|_{L^\infty(0,T;H^{3/2+\varepsilon})}$ such that for any $t \leqslant T$,*

$$\|\tilde{u}(t)\|^2 \leqslant e^{ct} \left\{ \|\tilde{u}(0)\|^2 + \int_0^T \|\tilde{f}(s)\|^2 \, ds \right\}.$$

Next consider the convergence of (1.8).

THEOREM 2. *If* $\gamma \geqslant \sigma \geqslant 3$ *and* $u \in C(0, T; H^{\sigma}_{(p)}(I))$, *then there exists a positive constant* C *depending on* $\|u\|_{L^{\infty}(0,T;H^{\sigma})}$ *such that for any* $t \leqslant T$,

$$\|u_C(t) - u(t)\| \leqslant CN^{1-\sigma}.$$

Now consider the generalized stability of (1.9). Suppose that u_c^k and the right term have, respectively, the error $\tilde{u}^k$ and $\tilde{f}^k \in \dot{v}_N$, then

$$\tilde{u}_t^k + J_C(\tilde{u}^k + \delta_1 \tau \tilde{u}_t^k, u_c^k + \tilde{u}^k) + J_C(u_c^k + \delta_1 \tau u_{ct}^k, \tilde{u}^k) + \tilde{u}_{xxx}^k + \delta_2 \tau \tilde{u}_{xxxt}^k = \tilde{f}^k. \tag{2.2}$$

Let

$$\rho^n = \|\tilde{u}^0\|^2 + \tau \sum_{k=0}^{n-1} \|\tilde{f}^k\|^2$$

and

$$E^n = \|\tilde{u}^n\|^2 + \alpha_0 \tau^2 \sum_{k=0}^{n-1} \|\tilde{u}_t^k\|^2.$$

THEOREM 3. *If* $\delta_1 = \delta_2 > \frac{1}{2}$, $\tau N^2 \leqslant d < \infty$, *and* $\varepsilon > 0$, *then there exists a positive constant* C *depending on* $|\!|\!| u_C |\!|\!|_{3/2+\varepsilon}$ *such that for all* $n\tau \leqslant T$,

$$E^n \leqslant c\rho^n e^{cn\tau}.$$

THEOREM 4. *If* $\delta_2 > \frac{1}{2}$, $\tau N^3 \leqslant d < \infty$, *and* $\varepsilon > 0$, *then there exist positive constants* C *and* δ *depending on* $|\!|\!| u_C |\!|\!|_{3/2+\varepsilon}$ *such that when* $\rho^{[T/\tau]} \leqslant \delta$ *and* $n\tau \leqslant T$,

$$E^n \leqslant c\rho^n e^{cn\tau}.$$

THEOREM 5. *If* $u \in C^1(0, T; H^1_{(p)}(I)) \cap C(0, T; H^{\sigma}_{(p)}(I))$ $(\gamma \geqslant \sigma \geqslant 3)$ *and* $\partial_t u \in H^1(0, T; L^2(I))$, *then there exists a positive constant* C *depending on* u *such that for any* $n\tau \leqslant T$,

(i) *if* $\delta_1 = \delta_2 > \frac{1}{2}$ *and* $\tau N^2 \leqslant d < \infty$, *then*

$$\|u_c^n - u^n\| \leqslant C\{\tau + N^{1-\sigma}\};$$

(ii) *if* $\delta_2 > \frac{1}{2}$, $\tau N^3 \leqslant d < \infty$, *and* $\tau + N^{1-\sigma}$ *suitably small, then*

$$\|u_c^n - u^n\| \leqslant C\{\tau + N^{1-\sigma}\}.$$

III. NUMERICAL RESULTS

EXAMPLE 1. Consider the Korteweg–de Vries equation

$$\partial_t \varphi + (1 + \varphi)\varphi_x + (\lambda^2/2)\varphi_{xxx} = 0,$$

which has the solitary wave

$$\varphi(x, t) = u_0 + a \operatorname{sech}^2[(a/b\lambda^2)^{1/2} (x - ct)], \tag{3.1}$$

where

$$c = 1 + u_0 + a/3.$$

Schamel and Elsässer [10] computed the above problem using both the spectral method (SM) and the pseudospectral method (PSM). The time integration used a stable fourth-order integration procedure. In Fig. 1, the solution with $\lambda = 10^{-2}$, $a = 0.2$, and

$$u_0 = -2\lambda(ba)^{1/2} \tanh[(a/24)^{1/2}/\lambda]$$

(so that $\int_0^1 \varphi(x, t)\, dx = 0$ for all times) is shown at $t = 1.25$ corresponding to 1000 time steps. Initially the soliton was centered at $x = -0.5$ (which corresponds to $x = 0.5$ because of periodicity). The SM solution agrees with the analytical solution φ (full curve), but the PSM solution φ_1 (dashed curve) has a large error due to the aliasing interaction.

By letting $u = 1 + \varphi$, we run the example using scheme (1.9) with $\delta_1 = \delta_2 = \frac{1}{2}$ and $\gamma = 5$. The solution φ_2 is shown in Fig. 1 (dotted curve). The error agrees with the convergence estimation in Section II (see Table I). Four time iterations per time step are required for the nonlinear term.

EXAMPLE 2. Consider the problem

$$\begin{aligned} &\partial_t u + \beta u u_x + \varepsilon u_{xxx} = 0, && 0 \leqslant x \leqslant 2,\ 0 < t \leqslant T, \\ &u(x, 0) = 3C \operatorname{sech}^2(Ax + D), && 0 \leqslant x \leqslant 2. \end{aligned} \tag{3.2}$$

The solution of (3.2) is

$$u(x, t) = 3C \operatorname{sech}^2(Ax - Bt + D),$$

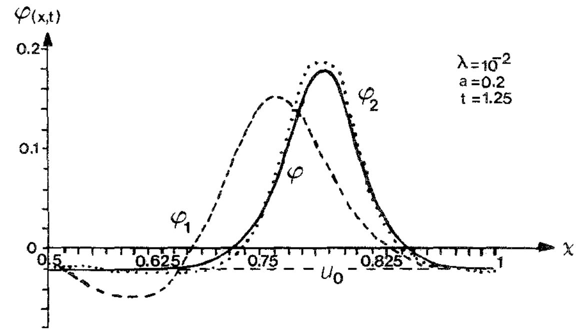

FIG. 1. The Korteweg–de Vries soliton at $t = 1.25$, φ is the solitary wave (3.1). φ_1 is computed with PSM in [10] (dashed line, $h = 1/32$). φ_2 is computed with (1.9) (dotted line, $h = 1/16$).

TABLE I

The L^2-Error and L^∞-Error of Scheme (1.9) at Time T, $\tau = 0.00125$, $h = 1/16$

T	L^2-error	L^∞-error
0.25	0.1970×10^{-2}	0.5728×10^{-2}
0.50	0.3298×10^{-2}	1.2137×10^{-2}
0.75	0.4469×10^{-2}	1.3146×10^{-2}
1.00	0.5596×10^{-2}	1.8146×10^{-2}
1.25	0.6754×10^{-2}	1.9735×10^{-2}

where

$$A = \tfrac{1}{2}(\beta C/\varepsilon)^{1/2}, \qquad B = \tfrac{1}{2}\beta C(\beta C/\varepsilon)^{1/2}.$$

For parameter values

$$C = 0.3, \qquad D = -6, \qquad \beta = 1, \qquad \varepsilon = 4.84 \times 10^{-4},$$

the calculation is carried out for $x \in [0, 2]$.

The L^∞-error of both the Hopscotch difference scheme (see [22]) and the scheme (1.9) with $\gamma = 10$ and $\delta_1 = \delta_2 = 0.6$ are shown in Tables II and III. The scheme (1.9) gives better results than the Hopscotch scheme.

IV. Some Lemmas

In order to prove the theorems in Section II, we need the following lemmas, the constants in which are independent of N and the function u and may be different in different cases.

Lemma 1 [16]. *If $0 \leqslant \mu \leqslant \sigma$ and $u \in H^\sigma_{(p)}(I)$, then*

$$\|p_N u - u\|_\mu \leqslant C N^{\mu - \sigma} |u|_\sigma, \tag{4.1}$$

$$\|p_N u\|_\sigma \leqslant C \|u\|_\sigma, \tag{4.2}$$

TABLE II

The Maximum Errors at Time T, $\tau = 0.025$, $h = 1/16$

T	Hopscotch scheme	Scheme (1.9)
0.25	0.1533	0.0343
0.50	0.2044	0.0458
0.75	0.2717	0.0627
1.00	0.2969	0.0797

TABLE III

The Maximum Errors at Time T, $\tau = 0.0005$, $h = 1/32$

T	Hopscotch scheme	Scheme (1.9)
0.25	2.231×10^{-2}	0.5997×10^{-3}
0.50	3.764×10^{-2}	1.228×10^{-3}
0.75	5.003×10^{-2}	1.873×10^{-3}
1.00	6.723×10^{-2}	2.637×10^{-3}

and if $\sigma > \frac{1}{2}$, then

$$\|p_c u - u\|_\mu \leqslant C N^{\mu - \sigma} |u|_\sigma, \tag{4.3}$$

$$\|p_c u\|_\sigma \leqslant C \|u\|_\sigma. \tag{4.4}$$

LEMMA 2 (Inverse Inequality) [16]. *If $0 \leqslant \mu \leqslant \sigma$ and $u \in v_N$, then*

$$\|u\|_\sigma \leqslant C N^{\sigma - \mu} \|u\|_\mu. \tag{4.5}$$

LEMMA 3. *If $u \in H^1(I)$, then*

$$\|u\|_{L^\infty} \leqslant C \|u\|^{1/2} \|u\|_1^{1/2}. \tag{4.6}$$

LEMMA 4. *For any $0 \leqslant \mu \leqslant \sigma \leqslant \gamma$, if $u \in v_N$, then*

$$\|Ru - u\|_\mu \leqslant C N^{\mu - \sigma} |u|_\sigma, \tag{4.7}$$

and if $u \in H^\sigma_{(p)}(I)$, then

$$\|R p_N u - u\|_\mu \leqslant C N^{\mu - \sigma} |u|_\sigma. \tag{4.8}$$

Proof. Let

$$u = \sum_{|k| \leqslant N} a_k e^{2\pi i k x},$$

then

$$\begin{aligned}
\|Ru - u\|_\mu^2 &\leqslant C' \sum_{|k| \leqslant N} (1 + |2\pi k|^{2\mu}) \left(\frac{|k|}{N}\right)^{2\gamma} |a_k|^2 \\
&\leqslant 2C' \sum_{|k| \leqslant N} |2\pi k|^{2\mu} |2\pi k|^{2\sigma} |2\pi k|^{-2\sigma} |a_k|^2 \\
&\leqslant C N^{2(\mu - \sigma)} \sum_{|k| \leqslant N} |2\pi k|^{2\sigma} |a_k|^2 \\
&= C N^{2(\mu - \sigma)} |u|_\sigma^2.
\end{aligned}$$

Proof of (4.8) follows from (4.7) and Lemma 1.

Now let

$$u = \sum_{|k| \leqslant N} a_k e^{2\pi i k x}$$

and

$$v = \sum_{|k| \leqslant N} b_k e^{2\pi i k x}.$$

Assume

$$a_{k+2N+1} = a_k, \qquad b_{k+2N+1} = b_k \tag{4.9}$$

and define the circle convolution

$$u * v = \sum_{|k| \leqslant N} \sum_{|l| \leqslant N} a_l b_{k-l} e^{2\pi i k x}. \tag{4.10}$$

It is easy to show that $u * v = v * u$.

LEMMA 5. *If $u, v \in v_N$ and $w \in \dot{v}_N$, then*

$$p_c(uv) = u * v, \tag{4.11}$$

$$(u * w, v) = (u, w * v). \tag{4.12}$$

Proof. It is sufficient to prove

$$u * v(x_j) = u(x_j)\, v(x_j), \qquad 0 \leqslant j \leqslant 2N.$$

Since

$$u * v(x_j) = \sum_{|l| \leqslant N} a_l e^{2\pi i l x_j} \sum_{|k| \leqslant N} b_{k-l} e^{2\pi i (k-l) x_j}$$

$$= \sum_{|l| \leqslant N} a_l e^{2\pi i l x_j} v(x_j) = u(x_j)\, v(x_j),$$

(4.11) follows. Then from (4.11) and (1.4),

$$(u * w, v) = (p_c(uw), p_c v) = (uw, v)_N$$

$$= (u, wv)_N = (u, w * v).$$

LEMMA 6. *For any $\varepsilon > 0$, if $u, v \in v_N$ and $w \in H_{(p)}^{3/2+\varepsilon}(I)$, then*

$$|(u_x * Rv, w) + (u * Rv_x, w)| \leqslant C_\varepsilon \gamma \|w\|_{3/2+\varepsilon} \|u\| \|v\|, \tag{4.13}$$

$$|(u_x * Ru, w) - (u * Ru_x, w)| \leqslant C_\varepsilon \gamma \|w\|_{3/2+\varepsilon} \|u\|^2, \tag{4.14}$$

where $R = R(\gamma)$ $(\gamma \geqslant 1)$ is defined by (1.6).

Proof. Assume that a_k and b_k are the coefficients of u and v respectively such that they have been extended as (4.9). Let

$$w = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x}$$

then

$$p_N w = \sum_{|k| \leqslant N} C_k e^{2\pi i k x}.$$

For any $|k| \leqslant N$ and $|l| \leqslant N$, define

$$\begin{aligned} r_{k,l} &= k - l + 2N + 1, && \text{if } k - l < -N, \\ &= k - l, && \text{if } |k - l| \leqslant N, \\ &= k - l - (2N + 1), && \text{if } k - l > N. \end{aligned}$$

Clearly $r_{-k,-l} = -r_{k,l}$. Since

$$\begin{aligned} (u_x * Rv, w) &= (u_x * Rv, p_N w) \\ &= 2\pi i \sum_{|k| \leqslant N} \bar{C}_k \sum_{|l| \leqslant N} \left(1 - \left|\frac{l}{N}\right|^{\gamma}\right) b_l r_{k,l} a_{k-l}, \end{aligned}$$

and

$$\begin{aligned} (u * Rv_x, w) &= (u * Rv_x, p_N w) \\ &= 2\pi i \sum_{|k| \leqslant N} \bar{C}_k \sum_{|l| \leqslant N} \left(1 - \left|\frac{l}{N}\right|^{\gamma}\right) l b_l a_{k-l}, \end{aligned}$$

then

$$\begin{aligned} I_1 &\equiv (u_x * Rv, w) + (u * Rv_x, w) \\ &= 2\pi i \sum_{|k| \leqslant N} \bar{C}_k \sum_{|l| \leqslant N} \left(1 - \left|\frac{l}{N}\right|^{\gamma}\right) (l + r_{k,l}) b_l a_{k-l} \\ &= 2\pi i \sum_{|k| \leqslant N} (1 + |k|) \bar{C}_k \sum_{|l| \leqslant N} f_{k,l} b_l a_{k-l}, \end{aligned} \tag{4.15}$$

where

$$f_{k,l} \equiv \left(1 - \left|\frac{l}{N}\right|^{\gamma}\right) (l + r_{k,l})/(1 + |k|).$$

Let

$$g_{k,l} \equiv \left(1 - \left|\frac{l}{N}\right|\right) (l + r_{k,l})/(1 + |k|),$$

then

$$|f_{k,l}| \leqslant \gamma\,|g_{k,l}|, \qquad \forall\,|k|,\,|l| \leqslant N.$$

In order to estimate $|g_{k,l}|$ for the three following cases, only the case of $0 \leqslant l < N$ is considered because $g_{-k,-l} = -g_{k,l}$.

Case 1. $|k-l| \leqslant N$, then

$$|g_{k,l}| = \frac{\left(1-\dfrac{l}{N}\right)|l+k-l|}{1+|k|} \leqslant 1.$$

Case 2. $k-l < -N$ and so $0 < N-l < -k = |k|$, then

$$|g_{k,l}| = \frac{(N-l)(2N+1+k)}{N(1+|k|)} \leqslant \frac{|k|\,(2N+1-|k|)}{(1+|k|)\,N} \leqslant 2.$$

Case 3. $k-l > N$. This is contrary to $l \geqslant 0$ and need not be considered. Therefore

$$|f_{k,l}| \leqslant \gamma\,|g_{k,l}| \leqslant 2\gamma, \qquad \forall\,|k|,\,|l| \leqslant N,$$

and from (4.15)

$$\begin{aligned}
|I_1| &\leqslant 4\pi\gamma \sum_{|k| \leqslant N} (1+|k|)|C_k| \sum_{|l| \leqslant N} |b_l|\,|a_{k-l}| \\
&\leqslant 4\pi\gamma \sum_{|k| \leqslant N} (1+|k|)|C_k| \left\{\sum_{|l| \leqslant N} |b_l|^2\right\}^{1/2} \left\{\sum_{|l| \leqslant N} |a_{k-l}|^2\right\}^{1/2} \\
&= 4\pi\gamma\,\|u\|\,\|v\| \sum_{|k| \leqslant N} (1+|k|)^{-(1/2+\varepsilon)}\,(1+|k|)^{3/2+\varepsilon}\,|C_k| \\
&\leqslant 4\pi\gamma\,\|u\|\,\|v\| \left\{\sum_{|k| \leqslant N} (1+|k|)^{-(1+2\varepsilon)}\right\}^{1/2} \left\{\sum_{|k| \leqslant N} (1+|k|)^{2(3/2+\varepsilon)}\,|C_k|^2\right\}^{1/2} \\
&\leqslant C_\varepsilon \gamma\,\|w\|_{3/2+\varepsilon}\,\|u\|\,\|v\|,
\end{aligned} \tag{4.16}$$

which completes the proof of (4.13).

On the other hand, since

$$\begin{aligned}
(u_x * Ru,\, w) &= (u_x * Ru,\, p_N w) \\
&= 2\pi i \sum_{|k| \leqslant N} \bar{C}_k \sum_{|l| \leqslant N} l\left(1-\left|\frac{r_{k,l}}{N}\right|^{\gamma}\right) a_l a_{k-l},
\end{aligned}$$

and

$$(u * Ru_x, w) = (u * Ru_x, p_N w)$$
$$= 2\pi i \sum_{|k| \leqslant N} \bar{C}_k \sum_{|l| \leqslant N} l\left(1 - \left|\frac{l}{N}\right|^\gamma\right) a_l a_{k-l},$$

then

$$I_2 \equiv (u_x * Ru, w) - (u * Ru_x, w)$$
$$= 2\pi i \sum_{|k| \leqslant N} \bar{C}_k \sum_{|l| \leqslant N} l\left(\left|\frac{l}{N}\right|^\gamma - \left|\frac{r_{k,l}}{N}\right|^\gamma\right) a_l a_{k-l}$$
$$= 2\pi i \sum_{|k| \leqslant N} (1 + |k|)\, \bar{C}_k \sum_{|l| \leqslant N} f_{k,l} a_l a_{k-l}$$

where

$$f_{k,l} \equiv l\left(\left|\frac{l}{N}\right|^\gamma - \left|\frac{r_{k,l}}{N}\right|^\gamma\right) \Big/ (1 + |k|).$$

Let

$$g_{k,l} \equiv l\left(\left|\frac{l}{N}\right| - \left|\frac{r_{k,l}}{N}\right|\right) \Big/ (1 + |k|),$$

then

$$|f_{k,l}| \leqslant \gamma\, |g_{k,l}|, \qquad \forall\, |k|, |l| \leqslant N.$$

In order to estimate $|g_{k,l}|$ for the three following cases, only the case of $0 \leqslant l \leqslant N$ is considered because $g_{-k,-l} = -g_{k,l}$.

Case 1. $0 \leqslant l \leqslant k$, then

$$|g_{k,l}| \leqslant \left|\, \left|\frac{l}{N}\right| - \left|\frac{r_{k,l}}{N}\right| \,\right| \leqslant 1.$$

Case 2. $k < l \leqslant N + k$ and so $0 < l - k = N$, then

$$|g_{k,l}| = \frac{l\,|l - (l-k)|}{N(1 + |k|)} = \frac{l\,|k|}{N(1 + |k|)} \leqslant 1.$$

Case 3. $l > N + k$, so $k - l < N$ and $0 \leqslant N - l < -k = |k|$, then

$$|g_{k,l}| \leqslant \frac{|l - (2N + 1 + k - l)|}{1 + |k|} = \frac{|2(N - l) + 1 - |k||}{1 + |k|}$$
$$\leqslant \max\left\{\frac{1 + 2(N - l)}{1 + |k|}, \frac{|k|}{1 + |k|}\right\} \leqslant 2.$$

Therefore

$$|f_{k,l}| \leqslant \gamma\, |g_{k,l}| \leqslant 2\gamma, \qquad \forall\, |k|,\, |l| \leqslant N.$$

Similarly using (4.16)

$$|I_2| \leqslant C_\varepsilon \gamma\, \|w\|_{3/2+\varepsilon} \|u\|^2,$$

which completes the proof of (4.14).

The next result follows immediately from (4.13) and (4.14).

LEMMA 7. *If* $\varepsilon > 0$ *and* $w \in H^{3/2+\varepsilon}_{(p)}(I)$, *then*

$$|(u_x * Ru, w)| \leqslant C_\varepsilon \gamma\, \|w\|_{3/2+\varepsilon} \|u\|^2, \qquad \forall u \in v_N. \tag{4.17}$$

V. THE PROOFS OF THE THEOREM

We now prove the theorems in Section II.

By (4.11), rewrite

$$J_C(u, v) = \tfrac{1}{3} u_x * Rv + \tfrac{1}{3}(u * Rv)_x.$$

Proof of Theorem 1. By taking the inner product of (2.1) with $2\tilde{u}$ and using (1.7)

$$\partial_t \|\tilde{u}\|^2 + 2(J_C(u_C, \tilde{u}), \tilde{u}) = 2(\tilde{f}, \tilde{u}).$$

Let $\varepsilon > 0$. Since

$$\begin{aligned} |(u_{cx} * R\tilde{u}, \tilde{u})| &= |(u_{cx} R\tilde{u}, \tilde{u})_N| \\ &\leqslant \|u_{cx}\|_{L^\infty} \|R\tilde{u}\|_N \|\tilde{u}\|_N \leqslant c_\varepsilon \|u_c\|_{3/2+\varepsilon} \|\tilde{u}\|^2, \end{aligned}$$

and

$$\begin{aligned} |((u_c * R\tilde{u})_x, \tilde{u})| &= |(u_c * R\tilde{u}, \tilde{u}_x)| \\ &= |(\tilde{u}_x * R\tilde{u}, u_c)| \leqslant c_\varepsilon \gamma\, \|u_c\|_{3/2+\varepsilon} \|\tilde{u}\|^2, \end{aligned}$$

then

$$|(J_c(u_c, \tilde{u}), \tilde{u})| \leqslant c_\varepsilon \gamma\, \|u_c\|_{3/2+\varepsilon} \|\tilde{u}\|^2. \tag{5.1}$$

Therefore

$$\partial_t \|\tilde{u}(t)\|^2 \leqslant c_\varepsilon \gamma\, \|u_c\|_{L^\infty(0,T;H^{3/2+\varepsilon})} \|\tilde{u}(t)\|^2 + \|\tilde{f}(t)\|^2,$$

and the conclusion of Theorem 1 follows.

Proof of Theorem 2. Let $w=p_N u$ and $\tilde{e}=u_c-w$ and using (1.1) and (1.8) produces

$$\partial_t \tilde{e}+J_c(\tilde{e}, w+\tilde{e})+J_c(w, \tilde{e})+\tilde{e}_{xxx}=f, \tag{5.2}$$

where

$$f=\tfrac{1}{3}\{p_N(uu_x+(u^2)_x)-p_c(w_x Rw)-(p_c(wRw))_x\}.$$

From Lemma 1 and Lemma 4

$$\begin{aligned}\|p_N(uu_x)-uu_x\| &\leqslant c' N^{1-\sigma}|uu_x|_{\sigma-1}\leqslant cN^{1-\sigma}\|u\|_\sigma^2,\\ \|uu_x-w_x Rw\| &\leqslant \|uu_x-uw_x\|+\|w_x u-w_x Rw\|\\ &\leqslant \|u\|_{L^\infty}|u-p_N u|_1+\|w_x\|_{L^\infty}\|u-Rp_N u\|\\ &\leqslant c(\|u\|_\sigma)\, N^{1-\sigma},\end{aligned}$$

and

$$\|w_x Rw-p_c(w_x Rw)\|\leqslant c' N^{1-\sigma}|w_x Rw|_{\sigma-1}\leqslant cN^{1-\sigma}\|u\|_\sigma^2,$$

so

$$\|p_N(uu_x)-p_c(w_x Rw)\|\leqslant c(\|u\|_\sigma)\, N^{1-\sigma}.$$

Similarly

$$\begin{aligned}\|p_N u^2-u^2\| &\leqslant cN^{-\sigma}\|u\|_\sigma^2,\\ \|u^2-wu\| &\leqslant cN^{-\sigma}\|u\|_{L^\infty}|u|_\sigma,\\ \|wu-wRw\| &\leqslant cN^{-\sigma}\|w\|_{L^\infty}|u|_\sigma,\\ \|wRw-p_c(wRw)\| &\leqslant cN^{-\sigma}\|u\|_\sigma^2.\end{aligned}$$

Since

$$\begin{aligned}\|p_N(u^2)_x-(p_c(wRw))_x\| &= |p_N u^2-p_c(wRw)|_1\\ &\leqslant cN\|p_N u^2-p_c(wRw)\|\leqslant c(\|u\|_\sigma)\, N^{1-\sigma},\\ \|f\| &\leqslant c(\|u\|_{L^\infty(0,T;H^\sigma)})\, N^{1-\sigma}.\end{aligned} \tag{5.3}$$

Also

$$\|\tilde{e}(0)\|\leqslant \|p_c u_0-u_0\|+\|u_0-p_N u_0\|\leqslant cN^{-\sigma}|u_0|_\sigma. \tag{5.4}$$

Finally, apply Theorem 1 to Eq. (5.2) and use Lemma 1 and the triangle inequality to complete the proof.

Proof of Theorem 3. Taking the inner product of (2.2) with $2\tilde{u}^k$ and $m\tau\tilde{u}_t^k$ $(m>0)$, respectively, produces

$$\begin{aligned}\|\tilde{u}^k\|_t^2-\tau\|\tilde{u}_t^k\|^2&+2\delta_1\tau(J_c(\tilde{u}_t^k, u_c^k+\tilde{u}^k), \tilde{u}^k)\\&+2(J_c(u_c^k+\delta_1\tau u_{ct}^k, \tilde{u}^k), \tilde{u}^k)+2\delta_2\tau(\tilde{u}_{xxxt}^k, \tilde{u}^k)\\&=2(\tilde{f}^k, \tilde{u}^k),\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}m\tau\|\tilde{u}_t^k\|^2&+m\tau(J_c(\tilde{u}^k, u_c^k+\tilde{u}^k), \tilde{u}_t^k)\\&+m\tau(J_c(u_c^k+\delta_1\tau u_{ct}^k, \tilde{u}^k), \tilde{u}_t^k)+m\tau(\tilde{u}_{xxx}^k, \tilde{u}_t^k)\\&=m\tau(\tilde{f}^k, \tilde{u}_t^k).\end{aligned} \tag{5.6}$$

Let $\varepsilon_1>0$. Combining (5.5) with (5.6) gives

$$\begin{aligned}\|\tilde{u}^k\|_t^2+\tau(m-1-\varepsilon_1)\|\tilde{u}_t^k\|^2+\sum_{i=1}^4 F_i&\\ \leqslant\|\tilde{u}^k\|^2+(1+\tau m^2/4\varepsilon_1)\|\tilde{f}^k\|^2,&\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}F_1&=2(J_c(u_c^k+\delta_1\tau u_{ct}^k, \tilde{u}^k), \tilde{u}^k),\\F_2&=m\tau(J_c(u_c^k+\delta_1\tau u_{ct}^k, \tilde{u}^k), \tilde{u}_t^k),\\F_3&=\tau(m-2\delta_1)(J_c(\tilde{u}^k, u_c^k+\tilde{u}^k), \tilde{u}_t^k),\\F_4&=\tau(m-2\delta_2)(\tilde{u}_t^k, \tilde{u}_{xxx}^k).\end{aligned}$$

Estimating $|F_i|$ using an argument similar to (5.1) produces

$$\begin{aligned}|F_1|&\leqslant c_\varepsilon|||u_c|||_{3/2+\varepsilon}\|\tilde{u}^k\|^2,\\|F_2|&\leqslant\tau\varepsilon_1\|\tilde{u}_t^k\|^2+\varepsilon_1^{-1}m^2\tau c(|||u_c|||_1)N^2\|\tilde{u}^k\|^2,\\|F_3|&\leqslant\tau\varepsilon_1\|\tilde{u}_t^k\|^2+\varepsilon_1^{-1}\tau(m-2\delta_1)^2N^2c(|||u_c|||_1)\{\|\tilde{u}^k\|^2+N\|\tilde{u}^k\|^4\},\\|F_4|&\leqslant\tau\varepsilon_1\|\tilde{u}_t^k\|^2+\varepsilon_1^{-1}\tau(m-2\delta_2)^2N^6\|\tilde{u}^k\|^2.\end{aligned}$$

Substituting the above estimation into (5.7) results in

$$\begin{aligned}\|\tilde{u}^k\|_t^2+\tau(m-1-4\varepsilon_1)\|\tilde{u}_t^k\|^2\leqslant c(|||u_c|||_{3/2+\varepsilon})\{&(1+\varepsilon_1^{-1}m^2\tau N^2)\|\tilde{u}^k\|^2\\&+\varepsilon_1^{-1}(m-2\delta_1)^2\tau N^2(\|\tilde{u}^k\|^2+N\|\tilde{u}^k\|^4)\\&+\varepsilon_1^{-1}(m-2\delta_2)^2\tau N^6\|\tilde{u}^k\|^2+\|\tilde{f}^k\|^2\}.\end{aligned} \tag{5.8}$$

By choosing $m>1$ properly and ε_1 suitably small such that

$$m-1-4\varepsilon_1=\alpha_0>0$$

and summing the formula (5.8) for all $0\leqslant k\leqslant n-1$, we get

$$\begin{aligned}\|\tilde{u}^n\|^2+\alpha_0\tau^2\sum_{k=0}^{n-1}\|\tilde{u}_t^k\|^2\leqslant\|\tilde{u}^0\|^2+c(|\!|\!|u_c|\!|\!|_{3/2+\varepsilon})\,\tau\sum_{k=0}^{n-1}\{&(1+\tau N^2)\|\tilde{u}^k\|^2\\ &+(m-2\delta_1)^2\,\tau N^2(\|\tilde{u}^k\|^2+N\|\tilde{u}^k\|^4)\\ &+(m-2\delta_2)^2\,\tau N^6\|\tilde{u}^k\|^2+\|\tilde{f}^k\|^2\}.\end{aligned}\tag{5.9}$$

Take $m=2\delta_1=2\delta_2>1$. From (5.9)

$$E^n\leqslant c(|\!|\!|u_c|\!|\!|_{3/2+\varepsilon})\{\rho^n+\tau\sum_{k=0}^{n-1}E^k\},$$

and the conclusion of Theorem 3 follows.

The following lemma is required to establish the generalized stability of (1.9) with $\delta_1\leqslant\frac{1}{2}$.

LEMMA 8 [21]. *If the following conditions hold,*

(i) *E^k is a nonnegative function and M, C, and ρ are nonnegative constants,*

(ii) *for any $n\tau\leqslant T$, if $\max_{0\leqslant k\leqslant n-1}E^k\leqslant M$, then*

$$E^n\leqslant\rho+c\tau\sum_{k=0}^{n-1}E^k,$$

(iii) $E^0\leqslant\rho\leqslant Me^{-cT}$,

then for any $n\tau\leqslant T$,

$$E^n\leqslant\rho e^{cn\tau}.$$

Proof of Theorem 4. Take $m=2\delta_2>1$ and let M be a positive constant. Assume $\max_{0\leqslant k\leqslant n-1}\|\tilde{u}^k\|\leqslant M$, then from (5.9)

$$E^n\leqslant c(|\!|\!|u_c|\!|\!|_{3/2+\varepsilon},M)\cdot\left\{\rho^n+\tau\sum_{k=0}^{n-1}E^k\right\}.$$

The conclusion follows from Lemma 9.

Proof of Theorem 5. Let $w^k=p_Nu^k$ and $\tilde{e}^k=u_c^k-w^k$. From (1.1) and (1.9)

$$\tilde{e}_t^k+J_c(\tilde{e}^k+\delta_1\tau\tilde{e}_t^k,w^k+\tilde{e}^k)+J_c(w^k+\delta_1\tau w_t^k,\tilde{e}^k)+(\tilde{e}_{xxx}^k+\delta_2\tau\tilde{e}_{xxxt}^k)=f^k\tag{5.10}$$

where

$$f^k = \partial_t w^k + \delta_2 \tau\, \partial_t w_t^k - w_t^k + p_N(u^k u_x^k) + \delta_2 \tau p_N(u^k u_x^k)_t - J_c(w^k + \delta_1 \tau w_t^k, w^k).$$

To estimate $|f^k|$, let $t_k = k\tau$, and intergrating by parts,

$$\partial_t u^k - u_t^k = -\frac{1}{\tau} \int_{t_k}^{t_{k+1}} (t_{k+1} - 5)\, \partial_t^2 u(x, s)\, ds,$$

and

$$\begin{aligned}\tau \sum_{k=0}^{n-1} \|\partial_t w^k - w_t^k\|^2 &= \tau \sum_{k=0}^{n-1} \|p_N(\partial_t u^k - u_t^k)\| \\ &\leqslant \frac{c'}{\tau} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |t_{k+1} - s|^2\, ds \int_{t_k}^{t_{k+1}} \|\partial_t^2 u(s)\|^2\, ds \\ &\leqslant c\tau^2 \|\partial_t^2 u\|^2_{L^2(0,T;L^2)}.\end{aligned}$$

Similarly

$$\begin{aligned}\tau \sum_{k=0}^{n-1} \|\delta_2 \tau\, \partial_t w_t^k\| &= \tau \sum_{k=0}^{n-1} \|\delta_2 \tau p_N(\partial_t u^k)_t\|^2 \\ &\leqslant c' \tau \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \cdot\, ds \int_{t_k}^{t_{k+1}} \|\partial_t^2 u(s)\|^2\, ds \\ &\leqslant c\tau^2 \|\partial_t^2 u\|^2_{L^2(0,T;L^2)}.\end{aligned}$$

From (5.3)

$$\|p_N(u^k u_x^k) - J_c(w^k, w^k)\| \leqslant c(\|u\|_{L^\infty(0,T;H^\sigma)})\, N^{1-\sigma}.$$

It is easy to show that

$$\|\delta_2 \tau p_N(u^k u_x^k)_t\| \leqslant c(\|u\|_{W^{1,\infty}(0,T;H^1)})\, \tau,$$

and

$$\|\delta_1 \tau J_c(w_t^k, w^k)\| \leqslant c(\|u\|_{W^{1,\infty}(0,T;H^1)})\, \tau,$$

whence

$$\left\{\tau \sum_{k=0}^{n-1} \|f^k\|^2\right\}^{1/2} \leqslant c\{\tau + N^{1-\sigma}\}.$$

Also from (5.4)

$$\|\tilde{e}^0\| \leqslant cN^{-\sigma} |u_0|_\sigma.$$

Finally, apply Theorem 3 or Theorem 4 to (5.10) and get the conclusions of Theorem 5 from Lemma 1 and the triangle inequality.

References

1. D. Gottlieb and S. A. Orszag, Numerical analysis of spectral methods, CBMS-NSF, Regional Conference series in applied mathematics, 26 (1977), Philadelphia.
2. J. Gazdag, *J. Comput. Phys.* **20**, 196 (1976).
3. J. P. Roache, *J. Comput. Phys.* **27**, 204 (1978).
4. K. Abe and O. Inoue, *J. Comput. Phys.* **34**, 202 (1980).
5. Y. Maday and A. Quarteroni, *Numer. Math.* **37**, 321 (1981).
6. O. H. Hald, *J. Comput. Phys.* **40**, 305 (1981).
7. Guo Ben-yu, *Acta Math. Sinica (Beijing)* **28**, 1 (1985).
8. Guo Ben-yu, *J. Comput. Math.* **1**, 353 (1983).
9. Guo Ben-yu and V. S. Manoranjan, *IMA JNA*, **5**, 307 (1985).
10. H. Schamel and K. Elsässer, *J. Comput. Phys.* **22**, 501 (1976).
11. J. Canosa and J. Gazdag, *J. Comput. Phys.* **23**, 393 (1977).
12. B. Fornberg and G. B. Whitham, *Philos. Trans. R. Soc. London* **289**, 373 (1978).
13. Y. Maday and A. Quarteroni, *SIAM J. Numer. Anal.* **19**, 761 (1982).
14. J. E. Pasciak, *SIAM J. Numer. Anal.* **19**, 142 (1982).
15. P. Cornille, *J. Comput. Phys.* **47**, 146 (1982).
16. H. O. Kreiss and J. Oliger, *SIAM J. Numer. Anal.* **16**, 421 (1979).
17. T. A. Zang and M. Y. Hussaini *in* "Proc. of Seventh International Conference on Numerical Methods in Fluid Dynamics," (Springer-Verlag, Berlin/New York, 1980).
18. J. E. Harvey, *Amer. J. Phys.* **49**, 747 (1981).
19. Cheng Min-teh and Chen Yung-ho, *Acta Sci. Natur. Univ. Pekinensis*, 411 (1956).
20. Kuo Pen-yu, *Sci. Sinica Ser. A* **25**, 702 (1982).
21. Guo Ben-yu, *Difference Methods for Partial Differential Equations* (Science Press, Beijing, 1986, in press).
22. I. S. Greig and J. L. Morris, *J. Comput. Phys.* **20**, 64 (1976).